GENERAL

# automorphic forms and l functions for the group gl nr

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## Automorphic Forms and L Functions for the Group GL(n, /! )

When number theorists first began to explore the deep connections between algebraic structures and complex analysis, they discovered a remarkable phenomenon: certain analytic functions—called **automorphic forms**—encode profound arithmetic information. Over the past half century, the study of these forms on the general linear group GL(n) has become a cornerstone of modern number theory, giving rise to powerful L functions that bridge representation theory, algebraic geometry, and the Langlands program. In this article we'll unpack the essentials of automorphic forms on GL(n), explore how their associated L functions are constructed, and examine the far reaching implications of this theory for mathematics and beyond.

## 1. What Are Automorphic Forms?

At its core, an automorphic form is a complex valued function defined on a group that remains invariant under the action of a discrete subgroup, up to a character. The classic example is the modular form on the upper half plane, invariant under the action of the modular group SL(2, !$). Generalizing this idea, an automorphic form on a reductive Lie group G is a smooth function on G that satisfies specific invariance, growth, and eigenvalue conditions under the action of a lattice 90 ", G. These functions can be viewed as eigenfunctions of a commutative algebra of invariant differential operators, making them amenable to harmonic analysis.

Key properties that distinguish automorphic forms include:

- Invariance under a discrete subgroup: f(;6r' óÒ öb†r' `or all ;2 ò , ó".
- Moderate growth at cusps, ensuring integrability over fundamental domains.
- Eigenfunctionality with respect to the center of the universal enveloping algebra of G.

These conditions allow automorphic forms to be decomposed into irreducible representations, which is the starting point for constructing associated L functions.

## 2. The Group GL(n) and Its Significance

The general linear group GL(n, ! ) consists of all invertible n × n real matrices. Its importance stems from several factors:

1. Universality: Every finite dimensional representation of a reductive group can be embedded into a GL(n) representation.
2. Rich structure: GL(n) possesses a well understood root system and Weyl group, facilitating explicit harmonic analysis.
3. Connection to arithmetic: The group GL(n) over global fields naturally appears in the study of automorphic representations of arithmetic groups.

Because of these attributes, GL(n) serves as a testing ground for general conjectures in the Langlands program, and many deep results in number theory have been proven in this setting.

## 3. Automorphic Forms on GL(n)

Automorphic forms on GL(n) are typically studied in two settings: the adelic formulation and the classical matrix valued function approach. In the adelic framework, one considers functions f /: /GL(n, /Ø5Ý8) /!' /! that are left invariant under GL(n, /! ) and right invariant under a maximal compact subgroup K /", /GL(n, /Ø5Ý8). The space of such functions decomposes into a direct sum of irreducible admissible representations, each called an **automorphic representation**.

In the classical setting, one examines functions on the generalized upper half plane for GL(n), defined as the quotient of GL(n, ! ) by a maximal compact subgroup. For example, a cusp form on GL(2) is a holomorphic function on the upper half plane that vanishes at cusps and satisfies a modular transformation law. The generalization to GL(n) involves higher dimensional analogues, such as Siegel modular forms for symplectic groups, but the underlying principles remain similar.

## 4. L Functions Associated to GL(n) Automorphic Forms

Every automorphic representation < of GL(n, Ø5Ý8) gives rise to an Euler product L function L(s, < 'à This function is defined as an infinite product over all places v of the global field (finite primes and infinite places), each factor reflecting the local behavior of < B c

$$\nL(s, /< ' óÒ ò õ÷·gÒ ôÅ÷b‡0, /< ÷b•ÆâB@

For non archimedean v, the local factor L_v(s, < ÷b• is a rational function in q_v^{-s} (where q_v is the size of the residue field) determined by the Satake parameters of < ÷`. At archimedean places, the local factors involve Gamma functions reflecting the analytic type of < ÷`. The resulting global L function captures both arithmetic and analytic aspects of < à

### 4.1. Satake Parameters and Euler Products

At almost all finite places v, the representation < ÷` is unramified, meaning it contains a non zero vector fixed by the maximal compact subgroup GL(n, Ø5Ý,_v). The eigenvalues of the Hecke operators acting on this spherical vector are encoded in the Satake parameters {; ÷·`,1}, /…, /; ÷·`,n}}. The local factor is then

$$\nL_v(s, /< ÷b' óÒ ò õ÷¶“Ó Õç¶çÒ òƒ ò!" ó±_{v,i} /q_v^{-s})^{-1}.\n$$

These factors assemble into the global Euler product, which converges absolutely in a right half plane and extends meromorphically to the complex plane.

### 4.2. Functional Equations

A hallmark of automorphic L functions is the existence of a functional equation relating L(s, /< ' Fò Âƒ ò!" ÷0, /\\tilde{< Ò'Â v†W \tilde{< Ò —2 F†R 6öçG agredient representation. In the GL(n) case, the completed L function 9²‡0, /< ' Ò ³(s, /< ' ôÂ‡0, /< ' 6 F

$$\n9²‡0, /< ' óÒ óµ(< •ÅÂÃ›(1 /" /s, /\\tilde{< Ò'ÅÆâB@

with ;RƒÀ) a complex number of absolute value 1, known as the root number. The gamma factor ;2‡0, < • is a product of Gamma functions determined by the archimedean components of < à This symmetry encodes deep duality properties and is central to the analytic study of L functions.

## 5. Key Properties: Euler Products, Functional Equations, and Analytic Continuation

The analytic behavior of L functions attached to GL(n) automorphic forms has been thoroughly investigated:

- Analytic Continuation: The completed L function 9²‡0, /< ' Pxtends to an entire function (except for possible poles at

s /= /0 or 1 for trivial representations).

- Growth Estimates: On vertical lines, 9²‡0, /< ’ 6 F—6f–W2 öÇ–æöÖ– Â prowth bounds, enabling the application of the Phragmén–Lindelöf principle.
- Zero Distribution: Generalized Riemann hypothesis for GL(n) automorphic L functions remains open, but numerous sub convexity results have been proven.

These properties are critical for applications such as bounding error terms in prime number theorems for arithmetic progressions and estimating Fourier coefficients of modular forms.

## 6. The Langlands Program and GL(n)

The Langlands program predicts a deep correspondence between automorphic representations of reductive groups and Galois representations. For GL(n), this correspondence is now largely established:

1. Local Langlands Correspondence: Associates to each irreducible admissible representation < ÷b öb tÂ†âÂ ôe÷b’ an n dimensional Frobenius semisimple Weil–Deligne representation < ÷b öb F†R Æö6 Â v Æö—2 proup G_{F_v}.
2. Global Langlands Correspondence: Relates automorphic representations < öb tÂ†âÂ ý*ƒ*]Ó…ôb’ Fò â F–ÖVç6–öæ Â continuous semisimple Galois representations < ¢ ôuôb ò " ôtÂ†âÂ ò "’à The local components of < Ö F6, F†R Æö6 Â 6 F °e parameters of < ÷`.

This bijective correspondence allows one to translate analytic properties of automorphic L functions into arithmetic properties of Galois representations and vice versa. In particular, the analytic continuation and functional equation of L(s, /< ’ Ö—'&÷" F†R Pxpected properties of Artin L functions attached to < à

### 6.1. Functoriality and Base Change

Functoriality, another pillar of the Langlands program, predicts that a homomorphism between L groups induces a transfer of automorphic representations. For GL(n), the most concrete instances involve:

- Base Change: Lifting automorphic forms from GL(n, /! ) to GL(n, /E) for a finite extension E/! .
- Automorphic Induction: The reverse process, inducing representations from a subgroup to GL(n).
- Symmetric Power Functoriality: Relating GL(2) cusp forms to GL(k) automorphic forms via symmetric power lifts.

These transfers preserve L functions: L(s, /< ’ óÒ ôÂ‡0, /< "’ `or the corresponding representations < æB À 2. Functoriality ha proven for many cases involving GL(n), providing a rich source of new automorphic representations and L functions.

## 7. Applications and Recent Advances

Automorphic forms on GL(n) and their L functions influence a broad spectrum of mathematical research:

- Prime Number Theorems in Arithmetic Progressions: The analytic properties of Dirichlet L functions, viewed as GL(1) automorphic L functions, yield distribution results for primes.
- Modularity Theorems: The proof of Fermat’s Last Theorem hinged on establishing modularity for elliptic curves, effectively GL(2) automorphic representations.
- Sub convexity Bounds: Recent work on sub convexity for GL(n) L functions has led to progress on equidistribution problems and quantum unique ergodicity.
- Explicit Reciprocity Laws: Through the Langlands correspondence, one can derive explicit relations between Fourier coefficients of automorphic forms and arithmetic invariants like class numbers.
- Automorphic Forms in Physics: Conformal field theory and string theory sometimes employ automorphic forms on higher rank groups to encode dualities.

In the last decade, breakthroughs such as the proof of the Sato–Tate conjecture for GL(2) and the establishment of

functoriality for symmetric powers up to degree 5 have expanded our understanding of the analytic behavior of automorphic $L$ functions.

## 8. Computational Aspects

Modern computational number theory relies heavily on explicit knowledge of automorphic forms and their $L$ functions. Key computational tasks include:


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