GENERAL

partial differential equations examples solutions

Published: September 3, 2026 | Index ID: #-

Partial Differential Equations: Examples, Solutions, and Practical Applications

Partial differential equations (PDEs) are the mathematical language that describes how physical quantities change over space and time. From the diffusion of heat in a metal rod to the propagation of waves on a string, PDEs capture the essence of dynamic systems. Yet, for many students and professionals, the subject can feel intimidating—especially when confronted with the phrase “partial differential equations examples solutions.” This article demystifies the topic by walking through classic examples, exploring common solution techniques, and illustrating how these equations model real-world phenomena. By the end, you’ll have a solid understanding of how to approach PDEs, recognize patterns in the equations, and apply the right method to find their solutions.

Why PDEs Matter in Science and Engineering

In the modern world, engineering and scientific problems rarely depend on a single variable. Instead, they involve multiple spatial dimensions and time. PDEs provide the framework to model such multi-variable systems. Here are a few domains where PDEs are indispensable:

- Heat Transfer – The heat equation predicts temperature distribution in solids.
- Fluid Dynamics – The Navier–Stokes equations govern the motion of fluids.
- Electromagnetism – Maxwell’s equations describe electric and magnetic fields.
- Quantum Mechanics – The Schrödinger equation governs wave functions.
- Structural Engineering – The wave equation models vibrations in beams and plates.

Because of their ubiquity, mastering PDEs opens doors to advanced research, innovative technology, and a deeper appreciation of the natural world.

Classifying Partial Differential Equations

Before diving into examples, it helps to classify PDEs. The classification is based on the type of the highest-order derivatives present:

1. Elliptic – Typically associated with steady-state problems (e.g., Laplace’s equation). They have no real characteristic curves.
2. Parabolic – Represent diffusion-type processes (e.g., heat equation). They have a single family of characteristic curves.
3. Hyperbolic – Model wave propagation (e.g., wave equation). They have two families of characteristic curves.

Each class has distinct solution strategies, which we’ll explore through examples.

Classic PDE Examples and Their Solutions

1. The Heat Equation (Parabolic PDE)

Equation: *u_t = ; SÇ7V#ç‡ƒÂ÷7V;#* where *u(x,t)* is temperature, ;is thermal diffusivity.

Suppose a metal rod of length L is initially heated uniformly at temperature T0. The rod’s ends are kept at zero

temperature for all time. The problem is to find the temperature distribution *u(x,t)* for 0 0.

Solution Strategy: Separation of variables.

1. Assume u(x,t) = X(x)T(t).
2. Substitute into the heat equation: X(x)T''(t) = ; ,rrrr‡,•B‡B’à
3. Divide by ; ,‡,•B‡B’ " T''(t)/(; T(t)) = X''''(x)/X(x) = -;² †6öç7F çB’à
4. Obtain two ODEs:

   T''(t) + ; » T(t) = 0 !’ T(t) = A e^{-; » t}.
5. X''''(x) + ;² ,‡,’ Ò v—F, &÷VæF y conditions X(0) = X(L) = 0.

The final solution captures how temperature decays over time and how higher modes dampen faster.

2. The Wave Equation (Hyperbolic PDE)

Equation: *u_{tt} = c² u_{xx}*, where *c* is wave speed.

Consider a plucked string of length L fixed at both ends. The initial displacement is *f(x)*, and the string is initially at rest (*u_t(x,0) = 0*).

Solution Strategy: D''Alembert’s formula or Fourier series.

1. Assume separation of variables: u(x,t) = X(x)T(t).
2. Obtain ODEs:

   T''''(t) + ;² 3'' T(t) = 0.
3. X''''(x) + ;² ,‡,’ Ò v—F, &÷VæF y conditions X(0) = X(L) = 0.

This solution shows how each Fourier mode travels back and forth at speed *c*, creating the familiar standing wave patterns.

3. Laplace’s Equation (Elliptic PDE)

Equation: *u_{xx} + u_{yy} = 0*, describing steady-state temperature or electrostatic potential.

Consider a rectangular plate of dimensions Lx × Ly held at different temperatures along its edges. We wish to find the temperature distribution *u(x,y)* inside the plate.

Solution Strategy: Separation of variables and Fourier series.

1. Assume u(x,y) = X(x)Y(y).
2. Substitute: X''''(x)Y(y) + X(x)Y''''(y) = 0 !’ X''''/X = -Y''''/Y = -;²à
3. Two ODEs:

   X'''' + ;² , Ò v—F, &÷VæF y conditions depending on the plate’s edges.
4. Y'''' - ;² Y = 0.

Laplace’s equation’s solution demonstrates how boundary conditions uniquely determine the steady-state solution inside a region.

4. The Burgers’ Equation (Nonlinear PDE)

Equation: *u_t + u u_x = ;Ò SÇ7V#ç‡ƒÂ÷7,*where ;is viscosity.

While more complex, Burgers’ equation serves as a simplified model for turbulence and shock wave formation. For

the inviscid case (;Ò Ò), the method of characteristics yields a simple solution that develops discontinuities (shocks) in finite time.

Solution Strategy: Method of characteristics.

1. Consider the characteristic ODE:
dx/dt = u(x,t), with initial condition u(x,0) = f(x).
2. Along a characteristic, du/dt = 0 (since ;Ò Ò ’Â 6ò R &VÖ –ç2 6öç7F çB Æöær F†R 7W ve.
3. Integrate to find the characteristic line: x = f(;â’ B ² ¾, where ;â Æ &VÇ2 F†R –æ—F– Â ÷6—F–öâà
4. Invert the relation to express ;â –â FW ms of x and t, then obtain u(x,t) = f(;â’à
5. When characteristics intersect, a shock forms. The shock location can be found by solving dx/d;â Ò à

For the viscous case (;Ò â), one can linearize using the Cole–Hopf transformation, converting Burgers’ equation into the heat equation and then applying the heat equation’s solution.

5. The Schrödinger Equation (Quantum PDE)

Equation: *i r #È/" t = - (sÇ7W ã#Â÷7W â ò &Ò’ #Ç7W ã#Â÷7W ãÈ/" x<sup>2 ;Ò â)+ V(x)t*

Description: Governs the evolution of quantum wave functions.

For a free particle (V = 0), the time-dependent Schrödinger equation reduces to a linear PDE similar to the heat equation but with a complex coefficient. The solution is a superposition of plane waves:

<,‡,ÇB’ Ò "² †²’ V’†·, Ò Ét) dk, where <’ Ò 'k2/2m.

The coefficients A(k) are determined by the initial wave function <,‡,Ã ’ F‡&÷Vv, `ourier transform.

General Solution Techniques

While each PDE has its own nuances, several overarching methods recur across examples:

1. Separation of Variables

Effective when the PDE and boundary conditions are linear and homogeneous. It reduces the PDE into a set of ordinary differential equations (ODEs).

2. Fourier Series and Transforms

Useful for problems on infinite or semi-infinite domains, or when boundary conditions are periodic. Fourier series express the solution as a sum of sines and cosines; Fourier transforms handle continuous spectra.

3. Method of Characteristics

Best suited for first-order PDEs and hyperbolic equations. It transforms the PDE into ODEs along curves (characteristics) in the domain.

4. Green’s Functions

Provides integral representations of solutions, especially for linear PDEs with inhomogeneous terms or complex geometries.

5. Numerical Methods

Finite difference, finite element, and

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